

Metric Operator For The Non-Hermitian Hamiltonian Model and Pseudo-Supersymmetry

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Abstract

We have obtained the metric operator $\Theta = \exp T$ for the non-Hermitian Hamiltonian model $H = \omega(a^\dagger a + 1/2) + \alpha(a^2 - a^{\dagger 2})$. We have also found the intertwining operator which connects the Hamiltonian to the adjoint of its pseudo-supersymmetric partner Hamiltonian for the model of hyperbolic Rosen-Morse II potential.

keyword: metric operator, pt symmetry, pseudo-super-symmetry

PACS: 03.65.w, 03.65.Fd, 03.65.Ge.

1 Introduction

For over a decade, quantum mechanics in complex domain, i.e. the field of \mathcal{PT} symmetric quantum mechanics[1] have attracted much more attention [2, 3, 4, 5, 6, 7, 8, 9, 10]. On the other hand, studies on the observations of \mathcal{PT} symmetry in experimental physics have been attracted much interest [11, 12, 13]. The Hamiltonian is \mathcal{PT} symmetric if $[H, \mathcal{PT}] = 0$ and coordinate, momentum operators acting on the Hilbert space are affected as \mathcal{PT} : $p \rightarrow p$, $x \rightarrow -x$, $i \rightarrow -i$ and linear operator \mathcal{P} commutes with the anti-linear operator \mathcal{T} , $[\mathcal{P}, \mathcal{T}] = 0$ where $\mathcal{P}^2 = \mathcal{T}^2 = 1$. The non-Hermitian quantum mechanics can be generalized to η -pseudo-Hermitian quantum mechanics within the inner products [14]. If \hat{A} is the pseudo-Hermitian operator, it satisfies $\eta A \eta^{-1} = A^\dagger$ where $\eta = \eta^\dagger$. The real spectrum of a non-Hermitian Hamiltonian is connected to a positive definite metric operator and quasi-Hermiticity[15] which is related to \mathcal{PT} symmetry is analyzed in the context of pseudo-Hermitian theory. Moreover, generalization of super-symmetric quantum mechanics within the pseudo-Hermiticity which is pseudo-supersymmetry has been studied by several authors [16, 17, 18, 19]. The non-Hermitian oscillator was first explored in [20] and in a series of papers the metric operator and algebraic properties are discussed [21, 22, 23, 24, 25], as well as it has generalized to the solvable models in quantum mechanics [26]. In this paper we have introduced a special form of the Swanson Hamiltonian in [20] which was studied in [27, 28] and we have obtained the metric operator. Moreover, the construction of metrics and Hermitian counterparts associated with $su(2)$ algebra can be found in [21]. Then, we have examined the Hamiltonian operator in a differential form and given the pseudo-Hermiticity of the Hamiltonian for a specific potential model called as complex hyperbolic Rosen Morse II.

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2 Pseudo-Hermitian Model

The reality of the spectrum is obtained with respect to a positive inner product $\langle \cdot, \cdot \rangle_+$ on the Hilbert space \mathcal{H} in which H is acting $\eta : \mathcal{H} \rightarrow \mathcal{H}$. Thus, the pseudo-Hermiticity of the Hamiltonian is given by

$$H^\dagger = \eta H \eta^{-1} \quad (1)$$

where η is positive definite, invertible operator that is related to \mathcal{C} operator by $\eta = \mathcal{C}\mathcal{P}$ [4] and it may be given as $\eta = e^{-Q}$, especially in perturbation theory [5, 6]. There may be some operators \mathcal{O} will also be observables, having real eigenvalues, such as

$$\mathcal{O}^\dagger = \eta \mathcal{O} \eta^{-1}. \quad (2)$$

If there is a similarity transformation

$$h = \Theta H \Theta^{-1} \quad (3)$$

where $\Theta = \sqrt{\eta}$, h is known as the Hermitian equivalent of H and this is called as the quasi-Hermiticity of H [15, 23]. Let us introduce a non-Hermitian model which is \mathcal{PT} symmetric

$$H = \omega(a^\dagger a + \frac{1}{2}) + \alpha(a^2 - a^{\dagger 2}) \quad (4)$$

where a, a^\dagger are standard boson operators

$$a = \sqrt{\frac{\omega}{2}} \hat{x} + i \frac{\hat{p}}{\sqrt{2\omega}}. \quad (5)$$

The spectrum of H is $E_n = (n + 1/2)\sqrt{\omega^2 + 4\alpha^2}$ which is the spectrum of h too. Now we use the symbol Θ for the metric operator and an ansatz for Θ which is

$$\Theta = \exp T \quad (6)$$

where the operator T can be taken as

$$T = \varepsilon a^\dagger a + \kappa(a^2 - a^{\dagger 2}) \quad (7)$$

where ε, κ are constants. If we use the transformations with Θ as given below

$$\Theta a \Theta^{-1} = \left(\cosh \theta - \frac{\varepsilon}{\theta} \sinh \theta \right) a + \frac{2\kappa}{\theta} \sinh \theta a^\dagger \quad (8)$$

$$\Theta a^\dagger \Theta^{-1} = \left(\cosh \theta + \frac{\varepsilon}{\theta} \sinh \theta \right) a^\dagger + \frac{2\kappa}{\theta} \sinh \theta a \quad (9)$$

where we use $\theta = \sqrt{\varepsilon^2 + 4\kappa^2}$. To find the Hermitian equivalent h , (3) can be expressed as below

$$h = \Theta H \Theta^{-1} = f_1(\varepsilon, \kappa)(a^\dagger a + \frac{1}{2}) + f_2(\varepsilon, \kappa)a^2 - f_3(\varepsilon, \kappa)a^{\dagger 2}. \quad (10)$$

Here, $f_i(\varepsilon, \kappa), i = 1, 2, 3$ are some functions depend on the Hamiltonian parameters. Now we can use

$$\Theta \Theta^{-1} \Theta a^\dagger \Theta^{-1} \Theta a \Theta^{-1} = \Theta a^\dagger a \Theta^{-1} \quad (11)$$

and $\Theta a \Theta^{-1}$, $\Theta a^\dagger \Theta^{-1}$ in order to obtain h . If we perform some algebra, we obtain

$$f_1(\varepsilon, \kappa) = \omega \cosh^2 \theta - \frac{\omega(\varepsilon^2 - 4\kappa^2) + 8\kappa\varepsilon\alpha}{\theta^2} \sinh^2 \theta \quad (12)$$

$$f_2(\varepsilon, \kappa) = \frac{2\kappa\omega}{\theta} L_- \sinh \theta + \alpha L_-^2 - \frac{4\kappa^2\alpha}{\theta^2} \sinh \theta \quad (13)$$

$$f_3(\varepsilon, \kappa) = -\frac{2\kappa\omega}{\theta} L_+ \sinh \theta + \alpha L_+^2 - \frac{4\kappa^2\alpha}{\theta^2} \sinh \theta \quad (14)$$

where

$$L_- = \cosh \theta - \frac{\varepsilon}{\theta} \quad (15)$$

$$L_+ = \cosh \theta + \frac{\varepsilon}{\theta}. \quad (16)$$

Then, because h must be Hermitian, considering (10) one can see that $f_2(\varepsilon, \kappa) = -f_3(\varepsilon, \kappa)$, thus we arrive at a condition which is

$$\frac{\tanh^2 \theta}{\theta^2} = \frac{\alpha}{\alpha(4\kappa^2 - \varepsilon^2) + 2\kappa\omega\varepsilon} \quad (17)$$

and this can be thought as the Hermiticity condition. We can use a parameter $z = \frac{2\kappa}{\varepsilon}$ as used in [22], then, ε is given as

$$\varepsilon = \pm \frac{1}{\sqrt{1-z^2}} \tanh^{-1} \sqrt{\frac{\alpha(1-z^2)}{\omega z - \alpha(1-z^2)}}. \quad (18)$$

Using (5), we can obtain

$$x = \Theta^{-1} \hat{x} \Theta = \left(\cosh \theta + \frac{2\kappa}{\theta} \sinh \theta \right) \hat{x} - i \frac{\varepsilon}{\omega \theta} \sinh \theta \hat{p} \quad (19)$$

$$p = \Theta^{-1} \hat{p} \Theta = \left(\cosh \theta - \frac{2\kappa}{\theta} \sinh \theta \right) \hat{p} + i \frac{\varepsilon \omega}{\theta} \sinh \theta \hat{x}. \quad (20)$$

Thus, the Hermitian equivalent h will have a form

$$h = \lambda_1(z) \hat{p}^2 + \lambda_2(z) \hat{x}^2 \quad (21)$$

where $\lambda_1(z)$ and $\lambda_2(z)$ can be found by using (19), (20) and ε as

$$\lambda_1(z) = \frac{\omega}{2} \left(U(1 + \sigma(z)) + V \left(\frac{2z(\omega + \alpha z)}{z\omega - 2\alpha(1 - z^2)} + 2\sqrt{\alpha}\sigma(z) \right) \right) \quad (22)$$

and

$$\lambda_2(z) = \frac{1}{2\omega} \left(U(1 - \sqrt{\alpha}\sigma(z)) + V \left(-\frac{4\alpha z}{\omega z - 2\alpha(1 - z^2)} + 2\sqrt{\alpha}\frac{\sigma(z)}{z} \right) \right) \quad (23)$$

where

$$\sigma(z) = z \frac{\sqrt{\alpha(1 - z^2) - z\omega}}{z\omega - 2\alpha(1 - z^2)} \quad (24)$$

$$U = \omega - \frac{4\alpha^2}{\omega z - 2\alpha(1 - z^2)} \quad (25)$$

$$V = \frac{z\omega(1 - \alpha)}{z\omega - 2\alpha(1 - z^2)} + \left(\frac{\omega}{2} - \frac{\alpha}{z}\right)\sqrt{\alpha}\sigma(z). \quad (26)$$

Finally, we have obtained $\Theta(z)$ as

$$\Theta(z) = \left(\frac{1 + \sqrt{\frac{\alpha(1-z^2)}{\omega z - \alpha(1-z^2)}}}{1 - \sqrt{\frac{\alpha(1-z^2)}{\omega z - \alpha(1-z^2)}}} \right)^{\pm \frac{1}{2\sqrt{1-z^2}}(a^\dagger a + \frac{\omega}{2}(a^2 - a^{\dagger 2}))} \quad (27)$$

If we take $z = 0$, then, Hermitian equivalent becomes

$$h = \frac{\omega}{2}(\omega - 2\alpha)\hat{p}^2 + \frac{1}{2\omega}(\omega - 2\alpha)\hat{x}^2 \quad (28)$$

that agrees with the results in [22] and [20].

3 Pseudo-Supersymmetry

Let us give (4) in terms of the differential operators using a, a^\dagger which can be generally given by

$$a = A(x)\frac{d}{dx} + B(x), \quad a^\dagger = -A(x)\frac{d}{dx} + B(x) - A(x)' \quad (29)$$

where $A(x), B(x)$ are real functions. Now, in terms of differential operators, (4) becomes

$$H = -\omega A(x)^2 \frac{d^2}{dx^2} + (4\alpha A(x)B(x) - 2\omega A(x)A(x)') \frac{d}{dx} - (\omega - 2\alpha)A(x)B(x)' - (\omega - 2\alpha)A(x)'B(x) + \omega B(x)^2 - \alpha(A(x)A(x))'' + A(x)'^2 + \frac{\omega}{2}. \quad (30)$$

The Hamiltonian can be mapped into a Hermitian operator form by using a mapping function ρ

$$h = \rho H \rho^{-1} \quad (31)$$

where

$$\rho = e^{-\frac{2\alpha}{\omega} \int dx \frac{B(x)}{A(x)}}. \quad (32)$$

Here we note that $H\Psi = \varepsilon\Psi$, $h\psi = \varepsilon\psi$, $\Psi = \rho^{-1}\psi$. So we can introduce operator h which is the Hermitian equivalent of H as

$$h = -\omega \frac{d}{dx} A(x)^2 \frac{d}{dx} + U_{eff}(x) \quad (33)$$

here $U_{eff}(x)$ takes the form

$$U_{eff}(x) = \frac{\omega}{2} - \omega(A(x)B(x))' - \alpha \left((A'(x))^2 + A(x)A''(x) \right) + \left(\omega + \frac{4\alpha^2}{\omega} \right) B^2(x) \quad (34)$$

where the primes denote the derivatives. Then (33) can be mapped into a Schrödinger-like form by using

$$\psi(x) = \frac{1}{A(x)}\Phi(x) \quad (35)$$

Hence, Schrödinger-like equation becomes [19]

$$-\Phi''(x) + \left(\frac{\omega/2 - \varepsilon}{\omega A^2(x)} - \frac{(A(x)B(x))'}{A^2(x)} + \frac{\omega^2 + 4\alpha^2}{\omega^2} \frac{B^2(x)}{A^2(x)} + \frac{\omega - \alpha}{\omega} \frac{A''(x)}{A(x)} - \frac{\alpha}{\omega} \frac{(A'(x))^2}{A^2(x)} \right) \Phi = 0. \quad (36)$$

If we use $A(x) = \cosh x$ and $B(x) = \delta \cosh x$ in above equation, we obtain the $V(x)$ which is seen as a part in (36) or $-\Phi''(x) + V(x)\Phi(x) = \lambda\Phi(x)$ as

$$V(x) = \delta^2 \frac{\omega^2 + 4\alpha^2}{\omega^2} - \left(\epsilon - \frac{1}{2} - \frac{\alpha}{\omega} \right) \sec^2 x + 2\delta \tanh x \quad (37)$$

where $\lambda = \frac{2\alpha}{\omega}$ is taken. This model is known as hyperbolic Rosen- Morse II potential [29]. Let us give a factorization procedure for the model in (37), so we write

$$H_p = -\frac{d^2}{dx^2} + V(x) = -\frac{d^2}{dx^2} + W(x)^2 - W(x)' \quad (38)$$

$$H = -\frac{d^2}{dx^2} + \bar{V}(x) = -\frac{d^2}{dx^2} + W(x)^2 + W(x)' \quad (39)$$

where $W(x)$ is the super-potential, H_p is the partner Hamiltonian of H which is [14, 16]

$$\eta H = H_p^\dagger \eta \quad (40)$$

where η is the intertwining operator and is a linear invertible operator in pseudo-Hermitian quantum theory. Let us give the super-potential that has a form as

$$W(x) = a \tanh x + b \quad (41)$$

and partner potentials are

$$V(x) = b^2 + a^2 - a(a+1) \sec^2 x + 2ab \tanh x \quad (42)$$

$$\bar{V}(x) = b^2 + a^2 + a(1-a) \sec^2 x + 2ab \tanh x \quad (43)$$

where

$$a = -\frac{4\alpha^2\delta^2 + 2\alpha\omega + \omega^2(1 + \delta^2 - 2\epsilon)}{2\omega^2} \pm \delta \frac{\sqrt{(\delta^2 - 4)\omega^4 + 8\alpha^2\delta^2\omega^2 + 16\alpha^4\delta^2}}{2\omega^2} \quad (44)$$

$$b = \frac{\delta}{a} \quad (45)$$

Let us choose the parameter b as $b \rightarrow ib$, then we have complex partner potentials

$$V(x) = -b^2 + a^2 - a(a+1) \sec^2 x + 2iab \tanh x \quad (46)$$

$$\bar{V}(x) = -b^2 + a^2 + a(1-a) \sec^2 x + 2iab \tanh x. \quad (47)$$

And the adjoint of H_p is

$$H_p^\dagger = -\frac{d^2}{dx^2} + V^\dagger(x) = -\frac{d^2}{dx^2} - b^2 + a^2 + a(a+1)\sec h^2 x - 2iab \tanh x \quad (48)$$

where $W(x)$ can be given as $W(x) = a \tanh x + ib$. Now, let us find the η_1 which intertwines H and H_p as $\eta_1 H = H_p \eta_1$. Then, we find η_1 as given below

$$\eta_1(x) = \frac{d}{dx} - a \tanh x + ib \quad (49)$$

On the other hand, we can give

$$\eta_2 H_p = H_p^\dagger \eta_2 \quad (50)$$

which means that H_p is η_2 -pseudo-Hermitian [16]. Then, we may give η_2 as

$$\eta_2 = \frac{d}{dx} - i\frac{a+1}{2b} \sec h^2 x \quad (51)$$

and finally one can give the operator η as

$$\eta = \left(\frac{d}{dx} - i\frac{a+1}{2b} \sec h^2 x \right) \left(\frac{d}{dx} - a \tanh x + ib \right). \quad (52)$$

The energy spectrum and the exact solutions of the (38) can be given as [30]

$$\lambda_n = -(a-n)^2 + \frac{a^2 b^2}{(a-n)^2} \quad (53)$$

$$\Phi_n = N(1 - \tanh x)^{\frac{c_1}{2}} (1 + \tanh x)^{\frac{c_2}{2}} P_n^{(c_1, c_2)}(\tanh x) \quad (54)$$

where $Re(c_1) > 0, Re(c_2) > 0$

$$n = 0, 1, \dots, n_{max}, \quad \frac{c_1 + c_2}{2} - a = -n, \quad c_{1n} = a - n + \frac{i\lambda}{a-n}, \quad c_{2n} = a - n - \frac{i\lambda}{a-n}. \quad (55)$$

H and H_p^\dagger are pseudo-super-partner Hamiltonians with the energy spectrum given above. In [30], the pseudo-norm for the wave-function was given in detail. One can look at [30] for obtaining the normalization constant N . Then, we gave probability density graphs using the Hamiltonian parameters in figures 1, 2, 3.

4 Conclusion

In conclusion, the metric operator is constructed for the model in (4). Taking $z = 0$ gives the parallel results with [22]. In [24], the authors studied a generalized quantum condition for the Swanson Hamiltonian and the symmetric nature of the Hamiltonian with respect to the parameters α and β . The model studied in this paper is the special case of the general frame related to the Hamiltonian in [24] when the parameters are taken as $\alpha = -\beta$. In our study we have used an operator (6) which is the metric operator where we take the operator T similar to the non-Hermitian Hamiltonian. The generalized Bogoliubov transformations using a general transformation operator and diagonalization gave the Hamiltonian in

Harmonic oscillator form [20], and in this paper after obtaining the metric operator we have discussed a special case (28) which is related to the harmonic oscillator. We have also studied the exact solvability of the model as giving the bosonic operators in terms of differential operators and according to the special choices of the functions in the first order differential operators we have given a special potential model. Using the concepts of pseudo-super-symmetry, we have obtained the pseudo-super-symmetric partners of the hyperbolic Rosen-Morse II potential in case of taking the one of the potential parameter b as ib . The operator η which leads to the real spectrum is obtained. The spectrum and wave-functions are given in terms of Hamiltonian parameters. We have given the probability density graphs for $n = 1$, $n = 2$ and $n = 8$. We have seen that the bound-states of the model depend on the parameters of the Hamiltonian which can also be seen from the graphs. In these graphs, the normalization constant is used as given in [30].

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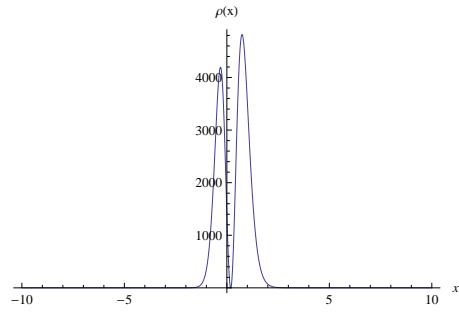


Figure 1: Graph of $\rho = |\Phi|^2$ for the parameters: $\alpha = 2; n = 1; \omega = 3; \delta = 10; \epsilon = 5$.

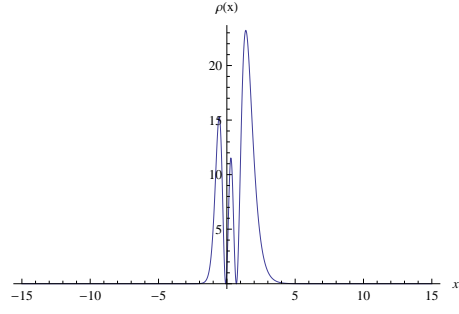


Figure 2: Graph of $\rho = |\Phi|^2$ for the parameters: $\alpha = 2; n = 2; \omega = 3; \delta = 10; \epsilon = 4$.

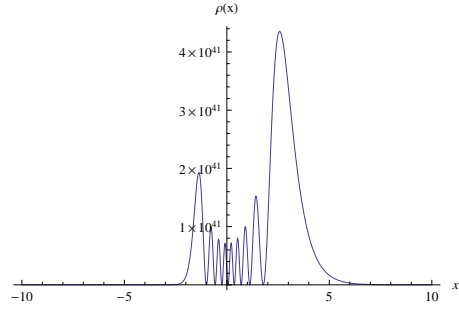


Figure 3: Graph of $\rho = |\Phi|^2$ for the parameters: $\alpha = 2; n = 8; \omega = 3; \delta = 10; \epsilon = 4$.